UDC 539.4

ON THE ANALYSIS OF THE MACROSCOPIC COEFFICIENTS OF STOCHASTICALLY INHOMOGENEOUS ELASTIC MEDIA

PMM Vol. 38, № 5, 1974, pp. 915-920

A. V. CHIGAREV
(Voronezh)

(Received August 2, 1973)

We study the application of the methods developed in the electrodynamics of stochastic media and adapted to the quantum field theory [1, 2], to the equations of the dynamics of an elastic medium. The method in question does not impose the usual restrictions of smallness on the magnitude of the fluctuations of the elastic moduli. Separation of the singular part of the Green tensor [1, 3, 4] and the introduction of new field variables, is equivalent to the summation of the quasi-static parts of the elastic moduli. Application of the discontinuous Weber-Schafheitlin integrals [5, 6] makes possible the accurate computation of the Born approximation. This brings to light various effects characteristic for the media possessing a structure [7]; when the dispersive properties can become discontinuous, resonant phenomena arise at the wavelengths comparable with the dimension of the structure. For a strongly isotropic medium [8] and an exponential correlation function, we give explicit expressions for the macroscopic elastic coefficients and for the eigenvalues of the operators.

The methods of computing the static macroscopic coefficients were developed and used by many authors (see e.g. [3, 4, 9, 10]). Under certain assumptions made about the mean stress-strain state and the texture of the medium, the operator relations connecting the average fields became, generally speaking, algebraic. When the dynamic effective parameters are computed, the relations appearing also have a non-local character and this complicates the dispersion equations considerably. In [11–14] the authors had computed, for the Born approximation, the macroscopic coefficients for the case of long and short waves when the spatial dispersion could be neglected.

1. The displacement vector u_i of a harmonic wave in an inhomogeneous medium, satisfies the equation

$$(\lambda_{ijkl} u_{k,l})_{,j} + \rho_0 \omega^2 u_i = 0 \qquad (1.1)$$

Here $\lambda_{ijkl}(r)$ depends on the spatial coordinates in a random manner, ρ_0 is the density, ω is the frequency, while the stresses and strains are connected by the Hooke's law.

Writing the equation (1.1) for a homogeneous medium with parameters λ_{ijkl}° and ρ_0, subtracting it from (1.1) and performing simple manipulations, we obtain

$$u_{i,m} = \overset{\circ}{u}_{i,m} - \int G_{in,mj}(r - r_1) \lambda_{njkl}'(r_1) u_{k,l}(r_1) dr_1 \qquad (1.2)$$

$$\lambda_{njkl}' = \lambda_{njkl} - \overset{\circ}{\lambda}_{njkl}$$

Here u_i° denotes the displacement field and $G_{in}(r - r_1)$ is the dynamic Green tensor in the homogeneous medium in question. Separating the integral in (1.2) into its singular and regular parts, we arrive at the new field quantities

$$E_{im} = \overset{\circ}{u}_{i,m} - \int G^{(R)}_{in,\,mj}(r - r_1)\, \gamma_{njst}(r_1)\, E_{st}(r_1)\, dr_1 \tag{1.3}$$

$$E_{im} = B_{imkl} u_{k,\,l}, \qquad \gamma_{njst} = \lambda'_{njkl} B^{-1}_{klst} \tag{1.4}$$

$$B_{imkl} = \delta_{ik}\delta_{ml} + G^{(s)}_{in.\,mj}\lambda'_{njkl}$$

where γ_{njst} is the polarization tensor, $G_{in}{}^{(s)}$ is the singular and $G_{in}{}^{(R)}$ the regular part of the Green tensor. The passage to the new variables E_{im} and γ_{njst} is equivalent to the process of summing an infinite series for the quasi-static part $L_{ijkl}(\omega, k)$ of the Fourier transform of the effective elastic moduli tensor.

Solving Eq. (1.3) by consecutive iterations, we obtain a series in powers of γ_{njst}. Averaging this series and utilizing the Feynman diagrams summation method, we can write the following Dayson-type [1, 2] equation for the average field $\langle E_{im}\rangle$:

$$\langle E_{im}\rangle = \overset{\circ}{u}_{i,m} + \iint G^{(R)}_{in,\,mj}(r - r_1)\, Q_{njst}(r_2 - r_1)\, \langle E_{st}(r_1)\rangle\, dr_1 dr_2 \tag{1.5}$$

The effective tensor $\overset{*}{\gamma}_{njst}$ is given by the relation

$$\langle \gamma_{njst} E_{st}\rangle = \Gamma_{njst}\langle E_{st}\rangle = \int \overset{*}{\gamma}_{njst}(r - r_1)\, \langle E_{st}(r_1)\rangle\, dr_1 \tag{1.6}$$

The kernel $\overset{*}{\gamma}_{njst}$ is connected with the mass operator Q_{njst} by the relation

$$\overset{*}{\gamma}_{njst}(r - r_1) = -Q_{njst}(r - r_1) \tag{1.7}$$

$$Q_{njst}(r_1 - r_2) = \langle \gamma_{njdl}(r_1)\, \gamma_{prst}(r_2)\rangle\, G_{dp,\,lr}(r_1 - r_2)$$

The series considered in the present method converge most rapidly, when

$$\langle \gamma_{njst}\rangle = 0 \tag{1.8}$$

The relations (1.8) yield a closed system of equations for determining the auxilliary coefficients in $\lambda_{ijkl}{}^{\circ}$ in terms of the moments λ_{ijkl}. The physical sense of Eq. (1.8) is clear, and the conditions of convergence of (1.8) do not impose any restrictions on the magnitude of the fluctuations of λ_{ijkl}.

Let us write the relations connecting the old and new variables

$$E_{im} = [\delta_{ik}\delta_{ml} + G^{(s)}_{in,\,mj}\lambda'_{njkl}]\; u_{k,l} = B_{imkl} u_{k,\,l} \tag{1.9}$$

$$\gamma_{njim} E_{im} = \lambda'_{njkl} B^{-1}_{klim} B_{imst} u_{s,\,t} = \lambda'_{njkl} u_{k,\,l} \tag{1.10}$$

Averaging (1.9) and (1.10) we obtain

$$\langle E_{im}\rangle = [\delta_{ik}\delta_{ml} + G^{(s)}_{in,\,mj}(\Lambda_{njkl} - \overset{\circ}{\lambda}_{njkl})]\, \langle u_{k,\,l}\rangle \tag{1.11}$$

$$\Gamma_{njim}\langle E_{im}\rangle = (\Lambda_{njkl} - \overset{\circ}{\lambda}_{njkl})\, \langle u_{k,\,l}\rangle \tag{1.12}$$

Substitution of (1.11) into (1.12) yields

$$\Gamma_{njim}[\delta_{ik}\delta_{ml} + G^{(s)}_{in,\,mj}(\Lambda_{njkl} - \overset{\circ}{\lambda}_{njkl})]\, \langle u_{k,\,l}\rangle = (\Lambda_{njkl} - \overset{\circ}{\lambda}_{njkl})\, \langle u_{k,\,l}\rangle \tag{1.13}$$

The tensor $\lambda_{njkl}{}^{*}$ is given by the relation

$$\langle \sigma_{ij}\rangle = \Lambda_{ijkl}\langle u_{k,\,l}\rangle = \int \overset{*}{\lambda}_{ijkl}(r - r_1)\, \langle u_{k,\,l}(r_1)\rangle\, dr_1$$

Let the system under consideration be statistically isotropic and homogeneous. We consider the average displacement fields of the form e^{ikr}. The relations (1.13) can

now be written as
$$D_{njim}\,[\delta_{ik}\delta_{ml} + G^{(s)}_{ip,mq}\,(L_{pqkl} - \overset{\circ}{\lambda}_{pqkl})] = L_{ijkl} - \overset{\bullet}{\lambda}_{ijkl} \qquad (1.14)$$

Here $D_{njim}\,(\omega, k)$ and $L_{njkl}\,(\omega, k)$ are the Fourier transforms of the kernels of the operators Γ_{njim} and Λ_{njkl}. Solving (1.14) for L_{njkl} we obtain

$$L_{stkl} = \overset{\circ}{\lambda}_{stkl} + M^{-1}_{stnj} D_{njkl} \qquad (1.15)$$
$$M_{pnqj} = \delta_{pn}\delta_{qj} - D_{njim} G^{(s)}_{ip,mq}$$

2. Let us turn our attention to finding the auxilliary coefficients. Let the medium defined by $\overset{\circ}{\lambda}_{ijkl}$ be isotropic and homogeneous, and let the inhomogeneous medium be isotropic

$$\overset{\circ}{\lambda}_{ijkl} = \lambda^{\circ}\delta_{ij}\delta_{kl} + \mu^{\circ}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad (2.1)$$
$$\lambda_{ijkl}\,(r) = \lambda\,(r)\,\delta_{ij}\delta_{kl} + \mu\,(r)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad (2.2)$$

Then in the formulas (1.4) we have

$$B_{imkl} = b_1\delta_{im}\delta_{kl} + b_2\delta_{ik}\delta_{ml} + b_3\delta_{il}\delta_{mk} \qquad (2.3)$$
$$b_1 = G_1^{(s)}\lambda' + 3G_2^{(s)}K', \qquad b_2 = 1 + G_1^{(s)}\mu', \qquad b_3 = G_1^{(s)}\mu'$$
$$B^{-1}_{imkl} = b_1^{-1}\delta_{im}\delta_{kl} + b_2^{-1}\delta_{ik}\delta_{ml} + b_3^{-1}\delta_{il}\delta_{mk}$$
$$b_1{}^{-1} = -b_1\,[(3b_1 + b_2 + b_3)\,(b_2 + b_3)]^{-1}, \quad b_2{}^{-1} = b_2\,(b_2{}^2 - b_3{}^2)^{-1}$$
$$b_3{}^{-1} = b_3\,(b_3{}^2 - b_2{}^2)^{-1}, \quad G_1{}^{(s)} = (3\lambda_0 + 8\mu_0)[15\mu_0\,(\lambda_0 + 2\mu_0)]^{-1}$$
$$G_2{}^{(s)} = -(\lambda_0 + \mu_0)[15\mu_0\,(\lambda_0 + 2\mu_0)]^{-1}$$
$$\gamma_{njst} = \gamma_1\delta_{nj}\delta_{st} + \gamma_2\,(\delta_{ns}\delta_{jt} + \delta_{nt}\delta_{js}) \qquad (2.4)$$
$$\gamma = K'\,[1 + K'\,(\lambda_0 + 2\mu_0)^{-1}]^{-1}, \quad \gamma = \gamma_1 + \frac{2}{3}\gamma_2,$$
$$K' = \lambda' + \frac{2}{3}\mu'$$
$$\gamma_2 = \mu'\,[1 + \mu'\,(6\lambda_0 + 15\mu_0)\,(15\lambda_0\mu_0 + 30\mu_0{}^2)^{-1}]^{-1}$$

The relations (1.8) give, in the isotropic case, two equations for determining λ_0, μ_0 and (K_0, μ_0) in terms of the moments λ and μ (K, μ). Assuming a definite law of distribution of λ, μ and K, we can calculate λ_0, μ_0 and K_0 in the manner analogous to that of [1]. For the composites and polycrystals we obtain self-consistent field equations.

Equations (1.8) and (2.4) can be solved by consecutive iterations without concretizing the mode of distribution. In the zero approximation we obviously have

$$\lambda_0 = \langle\lambda\rangle, \qquad \mu_0 = \langle\mu\rangle, \qquad K_0 = \langle K\rangle$$

In the first approximation

$$K_0^{(1)} = \langle K\rangle - \Lambda_{(0)}\,\langle K'^2\rangle + \Lambda^2_{(0)}\,\langle K'^3\rangle - \ldots \qquad (2.5)$$
$$\mu_0^{(1)} = \langle\mu\rangle - M_{(0)}\,\langle\mu'^2\rangle + M^2_{(0)}\,\langle\mu'^3\rangle - \ldots$$
$$\Lambda_{(0)} = \langle\lambda + 2\mu\rangle^{-1}, \quad M_{(0)} = 2\,\langle 3\lambda + 8\mu\rangle\,[15\,\langle\mu\rangle\,\langle\lambda + 2\mu\rangle]^{-1}$$

The values of λ_0, μ_0 and (K_0, μ_0) obtained in this manner, can now be substituted in all the above formulas.

3. Let us compute the second term of the formula (1.5). From the relations (1.5)–(1.7), with (2.1)–(2.4) in the Born approximation in $\overset{*}{\gamma}_{njkl}$ taken into account, it follows that the expression for the kernel $\gamma_{nj\gamma\delta}(r - r_1)$ has the form

$$\overset{*}{\gamma}_{nj\gamma\delta}(r-r_1) = \gamma_1^*(\rho)\,\delta_{nj}\delta_{\gamma\delta} + \gamma_2^*(\rho)\,(\delta_{n\gamma}\delta_{j\delta} + \delta_{n\delta}\delta_{j\gamma}) + \gamma_3^*(\rho)\,(\delta_{nj}n_\gamma n_\delta + \delta_{\gamma\delta}n_n n_j) + \gamma_4^*(\rho)\,(\delta_{n\gamma}n_j n_\delta + \delta_{n\delta}n_j n_\gamma + \delta_{j\delta}n_n n_\gamma + \delta_{j\gamma}n_n n_\delta) + \gamma_5^*(\rho)\,n_n n_j n_\gamma n_\delta \tag{3.1}$$

$$\rho = |r - r_1|, \qquad n_i = \rho_i\rho^{-1}$$

$$\gamma_m^*(\rho) = R(\rho)\left\{\frac{A_m(k_t\rho)}{\mu_0\rho^3}\, l^{ik_i\rho} + \frac{1}{\rho_0\omega^2\rho^5}\left[B_m(k_\alpha\rho)\, e^{ik_\alpha\rho}\right]_t^l\right\}$$

$$A_1 = 4z_t - k_t^2\rho^2, \quad B_1 = k_\alpha^4\rho^4 - 24ik_\alpha^3\rho^3 - 12z_\alpha, \quad A_2 = 2z_t$$

$$B_2 = 4(3z_\alpha + k_\alpha^2\rho^2), \quad A_3 = 2A_4, \quad A_4 = -(3z_t + k_t^2\rho^2)$$

$$B_3 = -2(30z_\alpha + z_\alpha^2k_\alpha^2\rho^2 + k_\alpha^4\rho^4 + 10ik_\alpha^3\rho^3), \quad A_5 = 0$$

$$B_4 = 4(6z_\alpha k_\alpha^2\rho^2 - 15z_\alpha - 5ik_\alpha^3\rho^3), \quad B_5 = 180k_\alpha^2\rho^2z_\alpha - 420z_\alpha + 4k_\alpha^4\rho^4 - 140ik_\alpha^3\rho^3, \quad z_\alpha = ik_\alpha\rho - 1$$

$$[f(k_\alpha\rho)]_t^l = f(k_l\rho) - f(k_t\rho), \quad k_\alpha^2 = \frac{\omega^2}{c_\alpha^2}, \quad c_t^2 = \frac{\mu_0}{\rho_0}$$

$$c_l^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \quad \langle\gamma_{njst}(r_1)\,\gamma_{\alpha\beta\gamma\delta}(r_2)\rangle = R(\rho)\, R_{njst}^{\alpha\beta\gamma\delta}$$

The eigenvalues of the operators Γ_{njim} and Λ_{njkl} can be found from the formulas

$$D_{nj\gamma\delta}(\omega, k) = \int \overset{*}{\gamma}_{nj\gamma\delta}(\rho)\, e^{-ik\rho}d\rho \tag{3.2}$$

$$L_{nj\gamma\delta}(\omega, k) = \int \overset{*}{\lambda}_{nj\gamma\delta}(\rho)\, e^{-ik\rho}d\rho$$

Passing in (3.2) to the spherical coordinates, we perform the integration over the angles. The integrand expressions are then transformed into the Bessel functions with half-integral indices and we obtain the discontinuous Weber-Schafheitlin and Sonine-Gegenbauer and Bailey integrals. Moreover, we regularize the integrals using the recurrence formulas for the Bessel functions. The computations are carried out for the correlation function $R(\rho) = R_0 \exp(-\rho_i / a_i)$ (a_i denote the correlation radii). The expressions for $D_{nj\gamma\delta}(\omega, k)$ and $L_{njkl}(\omega, k)$ also have the form of isotropic tensors

$$D_{nj\gamma\delta}(\omega, q) = D_1(\omega, q)\delta_{nj}\delta_{\gamma\delta} + D_2(\omega, q)(\delta_{n\gamma}\delta_{j\delta} + \delta_{n\delta}\delta_{j\gamma}) + D_3(\omega, q)(\delta_{nj}e_\gamma e_\delta + \delta_{\gamma\delta}e_n l_j) + D_4(\omega, q)(\delta_{j\delta}e_n e_\gamma + \delta_{j\gamma}e_n e_\delta + \delta_{n\delta}e_j e_\gamma + \delta_{n\gamma}e_j e_\delta) + D_5(\omega, q)e_n e_j e_\gamma e_\delta \tag{3.3}$$

$$e_i = q_i q^{-1}, \quad q^2 = k^2 + a^{-2}, \quad a^2 = a_i a_i$$

Let us write the explicit expressions for D_1 and D_2 corresponding to the case of a strongly-isotropic medium

$$D_m = \frac{R_0}{\mu_0}\left\{\frac{\Gamma_{mt}^{(1)}}{m} + \frac{2z_t^2}{15}(4\Gamma_{mt}^{(2)} + \Gamma_{mt}^{(3)}) + \frac{k_t^2\Gamma_{mt}^{(4)}}{2k} + \right. \tag{3.4}$$
$$\frac{4\Gamma_{mt}^{(5)}}{mz_t^2} + \Gamma_{mt}^{(6)} + \frac{4\mu_0}{\rho_0\omega^2}\left[\frac{k_\alpha{}^4\Gamma_{m\alpha}^{(4)}}{8k} + \frac{k_\alpha{}^2}{15}(\Gamma_{m\alpha}^{(7)} + \right.$$
$$5\Gamma_{m\alpha}^{(8)} + 5z_\alpha^{-2}\Gamma_{m\alpha}^{(9)} + 3\Gamma_{m\alpha}^{(10)} + \Gamma_{m\alpha}^{(11)} + 2\Gamma_{m\alpha}^{(12)} + 20z_\alpha^{-2}\Gamma_{m\alpha}^{(13)}) +$$
$$\frac{q^2}{315}(27\Gamma_{m\alpha}^{(5)} + z_\alpha^2\Gamma_{m\alpha}^{(14)} + 3\Gamma_{m\alpha}^{(15)} + 9\Gamma_{m\alpha}^{(16)} + 108\Gamma_{m\alpha}^{(17)} +$$
$$\left.\left. 4z_\alpha^{-2}\Gamma_{m\alpha}^{(18)}) + i\pi z_\alpha^{-1}q^{-4}(I_{m\alpha}^{(1)} + I_{m\alpha}^{(2)})\right]_t^l\right\}, \quad m = 1, 2$$

Depending on the relations between k_α and q, we obtain the following four different expressions for the coefficients of the polarization D_m :

1) $q < k_l, \quad z_\alpha = qk_\alpha^{-1}, \qquad \alpha = l, t$
   $\Gamma_{1l}^{(n)} = 0, n \neq 4, 7, 14, 17, \quad \Gamma_{1t}^{(p)} = 0, p \neq 1, 2, 3, 4, 7, 14, 17$
   $\Gamma_{2l}^{(n)} = 0, n \neq 14, 15, \quad \Gamma_{2t}^{(p)} = 0, p \neq 1, 3, 14, 15, \quad I_{m\alpha}^{(1)} = I_{m\alpha}^{(2)} = 0$
2) $q = k_l, \quad z_\alpha = k_l k_t^{-1}$
   $\Gamma_{2\alpha}^{(n)} = 0, \quad n \neq 1,3,14,15, \quad I_{m\alpha}^{(1)} = I_{m\alpha}^{(2)} = 0, \quad \alpha = l, t$
3) $k_l < q < k_t$
   $\Gamma_{1l}^{(n)} = 0, \quad n \neq 4,9,10,16, \quad \Gamma_{1t}^{(p)} = 0, \quad p \neq 1, 2, 4, 12, 14, 17$
   $\Gamma_{2l}^{(n)} = 0, \quad n \neq 11, 13, 16, 18, \quad \Gamma_{2t}^{(p)} = 0, \quad p \neq 1, 3, 15, \quad I_{mt}^{(1)} = I_{mt}^{(2)} = 0$
4) $k_t < q$
   $\Gamma_{1l}^{(n)} = 0, \quad n \neq 4, 8, 10, 16, \quad \Gamma_{1t}^{(p)} = 0, \quad p \neq 4, 5, 6, 8, 10, 16$
   $\Gamma_{2l}^{(n)} = 0, \quad n \neq 11, 13, 16, \quad \Gamma_{2t}^{(p)} = 0, \quad p \neq 5, 6, 11, 13, 16$
   $I_{1\alpha}^{(2)} = I_{2\alpha}^{(1)} = 0, \quad I_{1\alpha}^{(1)} \neq 0, \quad I_{2\alpha}^{(2)} \neq 0$
   $I_{m\alpha}^{(1)} = 5q^6 - 69q^4k_\alpha{}^2 + 211q^2k_\alpha{}^4 - 111k_\alpha{}^6$
   $I_{m\alpha}^{(2)} = 5q^6 - 9q^4k_\alpha{}^2 + 119q^2k_\alpha{}^4 - 61k_\alpha{}^6$

Here the coefficients $\Gamma_{m\alpha}^{(i)}$ are expressed, with (3.1) and (3.2) taken into account, in terms of the hypergeometric functions $F(\alpha, \beta, \gamma, z)$ in accordance with the Weber-Schafheitlin formulas [6]. The structure of the expressions for D_3, D_4 and D_5 is analogous to that of D_1 and D_2.

Let us note the following. Use of the discontinuous integrals in our computations makes it possible to obtain exact expressions for $D_m(\omega, q)$, and the relation connecting the wavelengths of the average and initial fields with the correlation radius, is the essential one. Study of the convergence of the hypergeometric functions shows, in the present case, that $F(\alpha, \beta, \gamma, z)$ converge wherever $\alpha + \beta - \gamma < 0$, and diverge at $z = 1$, if $0 \leqslant \alpha + \beta - \gamma < 1$. Thus the dispersion relations become discontinuous at $q = k_l$ and $q = k_t$. The asymptotic forms of the expressions for $D_m(\omega, q)$ are obtained, in the case when the spatial dispersion is neglected ($aq \ll 1$, $aq \gg 1$), from the above relations with $k_l, k_t < q$. With D_{njkl} known we can compute L_{njkl}, and the strongly isotropic field D_{njkl} has the corresponding strongly isotropic field L_{njkl}. Let us write the explicit expressions for L_1, L_2 and L in terms of D_1, D_2 and

D in the strongly isotropic case

$$L_2 = \mu_0 + \frac{D_2}{1 - 2G_1^{(s)}D_2}, \qquad L_1 = \lambda_0 + L - \frac{2}{3} L_2 \tag{3.5}$$

$$L = K_0 + D\left[\frac{1 - 2G_1^{(s)}D_2}{1 - 3\,(G_1^{(s)} + 3G_2^{(s)})\,D} + \frac{2G_1^{(s)}D_2}{1 - 2G_1^{(s)}\,D_2}\right]$$

The formulas (3.5) which give the eigenvalues of the effective elastic operators enable us to compute the velocities, the decay, the effective scattering cross section and other macroscopic coefficients of the medium in question.

REFERENCES

1. Ryzhov, Iu. A., Tamoikin, V. V. and Tatarskii, V. I., On the spatial dispersion of inhomogeneous media. ZhETF, Vol. 48, № 2, 1965.
2. Tatarskii, V. I., Propagation of Waves in Turbulent Atmosphere. Moscow, "Nauka", 1967.
3. Bolotin, V. V. and Moskalenko, V. N., Problem of determining the elastic constants of a microinhomogeneous medium. PMTF, № 1, 1968.
4. Fokin, A. G. and Shermergor, T. D., On computing the elastic moduli of heterogeneous media. PMTF, № 3, 1968.
5. Bateman, H. and Erdelyi, A., Higher Transcendental Functions. McGraw-Hill, 1955.
6. Bateman, H. and Erdelyi, A., Tables of Integral Transforms. McGraw-Hill, 1954.
7. Brillouin, L., Wave Propagation in Periodic Structures. McGraw-Hill, 1953.
8. Bolotin, V. V., Use of the Methods of the Theory of Probability and Reliability in the Analysis of Structures. M., Stroiizdat, 1971.
9. Lifshits, I. M. and Rozentsveig, L. N., On the theory of elastic properties of polycrystals. ZhETF, Vol. 16, № 11, 1946.
10. Khoroshun, L. P., On the theory of isotropic deformation of elastic media containing random inhomogeneities. Prikl. mekhan., Vol. 3, № 9, 1967.
11. Lifshits, I. M. and Parkhomovskii, G. D., On the theory of propagation of ultrasonic waves in polycrystals. ZhETF, Vol. 20, № 2, 1950.
12. Shuman, B. M., Propagation of elastic waves in a medium containing random inhomogeneities. Prikl. mekhan., Vol. 4, № 1, 1968.
13. Usov, A. A., Fokin, A. G. and Shermergor, T. D., On the theory of propagation of ultrasonic waves in polycrystals. PMTF, № 2, 1972.
14. Chigarev, A. V., Propagation of elastic waves in a stochastically inhomogeneous medium. Izv. Akad. Nauk SSSR, MTT, № 4, 1970.

Translated by L. K.